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PLANE TURBULENT COUETTE FLOW

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The problem on plane turbulent Couette flow for an incompressible liquid has been solved based on the anisotropic-turbulence model.

Let an incompressible Newtonian liquid located between two parallel planes be in the state of stationary turbulent flow due to the relative motion of the planes. Let the plane $y = 0$ be immobile and the plane $y = 2h$ move with a constant velocity U. The coordinate system is as follows: the x axis is in the direction of motion and the y axis is perpendicular to the planes (Fig. 1). Within the framework of the model of [1], the region between the planes is considered to be consisting of two layers of wall anisotropic turbulence the anisotropy in which is created by streamwise-extended Λ vortices [2] and the intermediate layer of isotropic turbulence. The boundaries of the wall turbulence at the planes are $y = y_1$ and $y = y_2$ ($y_1 < y_2$). Below we consider flow only in the wall regions $0 \le y \le y_1$ and $y_2 \le y \le 2h$.

Under the assumption that the liquid is incompressible and the flow is isothermal stationary and plane and with neglect of the mass forces, we seek the velocity u_i, the unit vector reference point n_i, and the pressure p in the form

$$u_x = u(y), \quad u_y = u_z = 0, \quad n_x = \cos\theta(y), \quad n_y = \sin\theta(y), \quad n_z = 0, \quad p = p(y). \tag{1}$$

The continuity equation is satisfied identically, and the equations of motion and state can be written as

$$\frac{d(\sigma_{xy} + \tau_{xy})}{dy} = 0, \quad \frac{d(-p + \sigma_{yy} + \tau_{yy})}{dy} = 0; \tag{2}$$

$$\frac{d\beta_{xy}}{dy} + g_x = 0, \quad \frac{d\beta_{yy}}{dy} + g_y = 0; \tag{3}$$

$$\sigma_{xy} = \sigma_{yy} = 0; \tag{4}$$

$$\tau_{xy} = \left(\mu_1 \sin^2\theta \cos^2\theta + \frac{1}{2}\mu_4\right)u', \quad \tau_{yy} = \mu_1 \sin^3\theta \cos\theta u'; \tag{5}$$

$$\beta_{xy} = \alpha_y \cos\theta - k_{22} \sin\theta \cos^2\theta\theta', \quad \beta_{yy} = \alpha_y \sin\theta - k_{22} \sin^2\theta \cos\theta\theta'; \tag{6}$$

$$g_x = \gamma\cos\theta + \alpha_y \sin\theta\theta', \quad g_y = \gamma\sin\theta - \alpha_y\cos\theta\theta' + k_{22}\sin\theta\theta'^2. \tag{7}$$

The quantities α_y, γ, μ_1, μ_4, and k_{22} in formulas (5)–(7) are considered to be constant.

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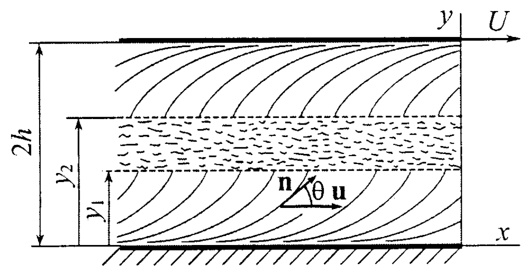

Fig. 1. Diagram of flow: the lower plane is immobile, while the upper plane moves with velocity U; $y = y_1$ and $y = y_2$; at each point of the curves, the vector reference point is directed tangentially to them.

The second equation of (2) determines the pressure distribution over the cross section of the flow in the wall regions

$$p(y) = \begin{cases} p(0) + \tau_{yy}(y) - \tau_{yy}(0), & 0 \le y \le y_1; \\ p(2h) + \tau_{yy}(y) - \tau_{yy}(2h), & y_2 \le y \le 2h. \end{cases} \quad (8)$$

The first equation of (2) simultaneously with the first equalities of (4) and (5) leads to the relation

$$\left(\mu_1 \sin^2\theta \cos^2\theta + \frac{\mu_4}{2}\right) u' = \pm\tau_w, \quad (9)$$

the (+) and (–) signs refer to the regions adjacent to the lower and upper planes respectively.

The substitution of expressions (6) and (7) into (3) leads to equations which coincide when $\gamma = 0$ and yield the equation for determination of the angle $\theta(y)$:

$$\sin\theta\cos\theta\theta'' + (1 - 3\sin^2\theta)\,\theta'^2 = 0. \quad (10)$$

Assuming the planes to be constant and disregarding the thickness of the laminar sublayer at the planes, we specify the boundary conditions

$$u(0) = 0, \quad u(2h) = U; \quad (11)$$

$$\sin\theta(0) = 0, \quad \sin\theta(2h) = 0. \quad (12)$$

Equalities (12) reflect the fact that vortices are extended streamwise at a solid wall [2].

Solving Eq. (10) with boundary conditions (12), we obtain

$$\cos\theta = \begin{cases} \pm(1 - 3by)^{1/3}, & 0 \le y \le y_1; \\ \mp[1 - 3b(2h - y)]^{1/3}, & y_2 \le y \le 2h. \end{cases} \quad (13)$$

The constant b is determined by the equality [1]

$$b^2 = \sin^2\theta_0 \cos^4\theta_0 \theta_0'^2, \quad (14)$$

where θ_0 and θ_0' are the angle θ and its derivative at the upper boundary of the vortex layer adjacent to each plane. In view of the symmetry the constant b is equal for both regions. The combination of signs (+, –) in formulas (13) corresponds to $\theta = 0$ on the immobile plane and $\theta = \pi$ on the moving plane, while the combination (–, +), conversely, corresponds to $\theta = 0$ on the moving plane and to $\theta = \pi$ on the immobile plane (Fig. 1).

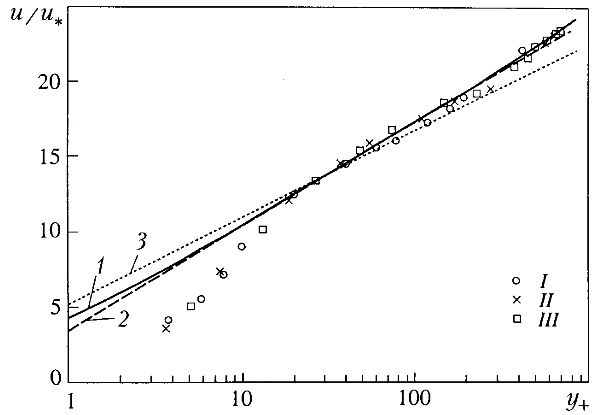

Fig. 2. Dimensionless velocity u/u_* vs. dimensionless coordinate $y_+ = yu_*/\nu$ in the region $0 \le y \le y_1$: 1) calculation from formulas (15)–(17); 2) the same, from the formula $u/u_* = 2.95 \ln y_+ + 3.44$; 3) the same, from the formula $u/u_* = 2.55 \ln y_+ + 5.2$; points, experiment [3] (I, II, and III, see Table 1).

TABLE 1. Experimental Data [3] and Values of the Parameters Employed for Obtaining the Curves in Fig. 2

Experiment	U, m/sec	h, mm	u_*, m/sec	b, 1/m	μ_1, kg/(m·sec)	μ_4, kg/(m·sec)	q
I	12.84	22	0.293	3.56	0.017	$3.81 \cdot 10^{-6}$	1.000056
II	12.84	33	0.282	3.40	0.017	$3.88 \cdot 10^{-6}$	1.000056
III	17.08	33	0.363	3.52	0.021	$3.78 \cdot 10^{-6}$	1.000045

Integration of Eq. (9) with account for the solution (13) and boundary conditions (11) yields the sought velocity profiles:

$$u(y) = \begin{cases} \Phi(t), & t = (1 - 3by)^{1/3}, \quad 0 \le y \le y_1; \\ U - \Phi(t), & t = [1 - 3b(2h - y)]^{1/3}, \quad y_2 \le y \le 2h; \end{cases} \tag{15}$$

$$\Phi(t) = \frac{2u_*}{\kappa(2q^2 - 1)} \left[\sqrt{q^2 - 1} \left(\arctan \frac{t}{\sqrt{q^2 - 1}} - \arctan \frac{1}{\sqrt{q^2 - 1}} \right) + \frac{q}{2} \left(\ln \frac{q - t}{q + t} - \ln \frac{q - 1}{q + 1} \right) \right], \tag{16}$$

$$\alpha = \frac{\mu_4}{2\mu_1}, \quad 2q^2 = 1 + \sqrt{1 + 4\alpha}, \quad q > 0, \quad u_* = \sqrt{\frac{\tau_w}{\rho}}, \quad \kappa = \frac{2\mu_1 b}{\rho u_*}. \tag{17}$$

Near the planes, for example, the plane $y = 0$, when $3by \ll 1$, this solution yields the logarithmic law of the wall

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln y_+ + C, \quad C = \frac{1}{\kappa} \ln \frac{b\nu}{(q - 1) u_*}. \tag{18}$$

The quantities κ and C that are usually considered as empirical constants are expressed here by the parameters of the medium and the flow ρ, ν, μ_1, b, q, and u_*, which does not prevent them from remaining constant.

Figure 2 gives the experimental points [3] and the graphs of three velocity profiles in the region $0 \le y \le y_1$ which have been calculated from formulas (15)–(17) under these experimental conditions. The working fluid is air. The density is $\rho = 1.21$ kg/m^3 and the kinematic viscosity is $\nu = 1.486 \cdot 10^{-5}$ m^2/sec. The parameters that change in the experiments and the design parameters of the model are given in Table 1. For $y_+ > 20$ all the experimental points are located near curve 1, which represents the image of three difficult-to-differentiate curves (15)–(17) calculated with allowance for the data for experiments I–III. Curve 2 represents the graph of the function (18) with average values of $\kappa = 0.339$ and $C = 3.44$. Clearly, this curve is similar to the calculated curves 1 and it better approximates the points than the function (18) with constants $\kappa = 0.39$ and $C = 5.2$ from [3] (curve 3).

From the considered solution it is clear that the anisotropic-turbulence model used enables one to easily describe wall turbulent flow, employing only physically clear parameters of the moving medium and the boundary conditions. Some of the known empirical constants can be expressed by the parameters of the model and the flow.

NOTATION

b, integration constant determined by formula (14), 1/m; C, constant in formula (18); g_i, density of the internal generalized force, kg/(m·sec^2); h, half the distance between parallel planes, m; k_{22}, determining constant of the model, kg·m/sec^2; **n** and n_i, unit vector reference point and its projections onto the x, y, and z axes; p, pressure, Pa; q, constant determined by the second formula of (17); u, longitudinal local velocity, m/sec; U, velocity of the moving plate, m/sec; **u** and u_i, vector of the local velocity of the liquid and its projections onto the x, y, and z axes, m/sec; u_*, dynamic velocity, m/sec; x, y, z, Cartesian coordinates; y_1 and y_2; coordinates determining the boundaries of the wall layers at the immobile and moving planes; $y_+ = (u_* y)/\nu$, dimensionless coordinate; α, constant determined by the first formula of (17); α_y, projection of the arbitrary vector function α_i onto the y axis, kg/sec^2; β_{ij}, generalized stresses, kg/sec^2; γ, arbitrary function involved in Eq. (17), kg/(m·sec^2); θ, angle of inclination of the vector reference point to the x axis; κ, von Kármán constant; μ_1 and μ_4, determining constants of the model, kg/(m·sec); ν, kinematic viscosity, m^2/sec; ρ, density, kg/m^3; σ_{ij}, stresses caused by the presence of the structure in the medium, Pa; τ_{ij}, viscous stresses, Pa; τ_w, modulus of tangential stress on the wall, Pa. Superscripts and subscripts: ($'$), derivative with respect to the coordinate y; w, wall.

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